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A Morphology of Teacher Discourse in the Mathematics Classroom

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Abstract: Discourse in mathematics classrooms is surprisingly complex and both student and teacher mathematical discourse contain distinct, identifiable elements. Student discourse is necessarily focused on understanding concepts and solving mathematical problems. Teacher discourse contains some of these same elements, but when examined critically it gives rise to major distinctions. Teacher discourse is directed at improving student understanding and also the logistics of the classroom, and thus is often meta-mathematical in nature. We shine a light on the *tactics* teachers use which are part of meta-mathematical discourse such as re-voicing, redirecting, questioning, and clarifying. Contrasts are explored between student and teacher discourse.

Key words: Discourse morphology; Discourse tactics; Discourse tools; Meta-mathematical discourse; Student discourse; Teacher discourse

Introduction

The purpose of this paper is to first lay out and distinguish among different types and aspects of teacher discourse in mathematics, and then to contrast these with *student* discourse. Many papers focus in detail on one particular aspect (gestures, questioning, listening, responding) of teacher discourse rather than laying out all the many facets of it (e.g., Ferreira & Presmeg, 2004; Davis, 1997; Voigt, 1985). These authors do not emphasize the differences between types of *student* and *teacher* discourse. In this paper, particular attention will be afforded the distinctive differences between student discourse and the multifaceted teacher discourse in mathematics classrooms.

This paper is not intended to create a theory of discourse as such but to provide a coherent synthesis of extant theory with concrete examples to illuminate theoretical constructs. The examples provided come from the transcribed discourse in a 9th grade mathematics course in which the teacher introduces the class to a non-Euclidean metric, namely the Manhattan metric. The classifications suggested here could be altered and renumbered, combined or subdivided at will. However for the sake of discussion, it is useful to identify groupings that naturally occur in mathematical teacher discourse. In this way, teachers can weigh how much time they spend in the various stages of teaching, and evaluate their own classrooms. They can see where they might say more or clarify their lessons with probing questions that are better directed at the students' growing content knowledge.

There are substantial differences in types and forms of discourse encountered, with teacher discourse playing a central role in establishing classroom climate, engagement of students, and in creating opportunities for student improvement and reflection. This paper is not an attempt to carry out a complete meta-analysis of the subject of discourse in mathematics. By creating this brief analysis of types, the daunting task of a complete review may be shown to be both necessary and facilitated at some future time. Even though this is not a report of a particular empirical research study, it is informed by the authors' many years in university and secondary mathematics classrooms, focused observations of other teachers, including reviewing video transcripts, and teaching a variety of professional development courses, seminars and symposia for mathematics teachers of all levels. This paper will aid researchers and mathematics teachers who wish to focus on mathematical discourse research as well as those who just want to do a better job of teaching by providing a focus for reflection on discourse. The effects on student engagement, participation and learning of what teachers say and how they say it will be better understood through this morphological lens.

Background

It was once believed that students' minds were like sponges just waiting to absorb information downloaded from the teacher. Students in these passive environments tended to conform by listening passively and quietly at their desks, learning from exercises without engaging directly in discourse with the teacher, other students or the subject. Prior to recent reform movements in mathematics, much mathematics instruction was monologic in nature, that is, the teacher did most if not all of the talking. Studies of this type of discourse revealed it to be necessarily limited and one dimensional, flowing uni-directionally from teacher to student (Wertsch & Toma, 1995; Knuth & Peressini, 2001). The ways in which this information was imparted or received were not held to be particularly important. Teachers who were

competent in mathematics, quite apart from any pedagogical preparation, were considered to be appropriate candidates to fill important teaching positions.
The advent of reform ideas in mathematics education, however, together with an emphasis on constructivism, brought with it fresh ideas of how to engage students more effectively in the classroom, using small group work, open ended tasks, guided discovery, modeling and similar innovative ideas. Now suddenly the classroom was rich in dialogic discourse, as students were being encouraged to think and speak.

Discourse, as a field of study, has been the focus of renewed interest among mathematics educators in the last 20 years. What teachers say and how they say it is important and worth this renewed interest. Simply stated, the language used in the classroom has a significant influence on how and what students learn. Many researchers (e.g. Ferreira & Presmeg, 2004; Sfard, 2000a, 2000b; Goldin, 2000; Davis, 1997; McLeod, 1994) have concentrated on describing various aspects of discourse that occur in mathematics classrooms, sometimes focusing on a particular form such as questioning or listening. Mathematics educators (e.g., Knuth & Peressini, 2001) rightly focused on this new dialogic discourse occurring among students, and between students and teachers. However, this student discourse has a different nature than the teacher's. In this paper we turn the spotlight back on the teacher to better understand what discourse moves might be important in increasing student learning in the classroom, and to contrast these types of discourse.

Lerman (2001) equates learning mathematics and learning to think mathematically with learning to speak mathematically (p. 107). Learning school mathematics, he claims, is "nothing more than initiation into the practices of school mathematics, hence the central role of the initiator, the teacher" (p. 107). He also points to the power of the structure of discourse itself in placing participants in positions of powerfulness and powerlessness, as well as their own personal histories (p. 105). He suggests that researchers need to find a way to incorporate both the macro and micro events in learning into a cohesive whole. It is important for teachers to bear in mind the power and influence of this social role when preparing and participating in a classroom episode. Sfard (1991, 2000a, 2000b, 2001) has written extensively about discourse in mathematics and in particular on focal analysis (analysis of the focus of the discourse) and its role in analyzing the process of mathematics discourse. She has observed three distinct foci existing in any analysis of mathematical discourse – the pronounced focus, the attended focus and the intended focus. She defines them as follows: the pronounced focus – the words used by the interlocutor; the attended focus - what the interlocutor is looking at, listening to, etc.; and the intended focus - the interlocutor's intention in contributing to the discourse. All this is about what is happening to the ball of the lesson objective as it

is hit back and forth across the net between students and teacher. While this particular unit of analysis has proved helpful in examining and analyzing discourse, it applies more aptly as a descriptor for the process of learning, rather than any characterization of the *types* of discourse as discussed here. Sfard's analysis will be referred to again later.

Davis (1997) suggests that the art of listening is at least as important as the art of discourse in the mathematics classroom. He explores three categories of listening – *evaluative*: listening for a particular, preconceived "right" answer or explanation, or listening in order to respond; *interpretive:* listening for sense-making, and for student understanding; and *hermeneutic*: listening to the speaker as a prelude to and as a component of a negotiation for meaning in a situation. In the authors' experience, listening is one of the most important – if not the most important – skills a teacher can develop. This art integrates naturally with a focus on types of discourse. A teacher must listen to his students in order to sustain effective discourse.

Davis' (1997) emphasis on listening is an important piece of the discourse puzzle, as is his connection of types of listening with teachers' conceptions of the nature of mathematics. He provides evidence that how teachers conceive of mathematics (and their empathy relating to what it takes different kinds of students to learn new mathematical concepts –cognitive demand) affects how they listen to students in a mathematics class. Beyond that, it is not only essential to be a good listener, but one must be *perceived* to be a good listener to gain the optimum participation by students. Students need to feel that what they say matters. Active discourse cannot occur unless teachers practice active interpretive or hermeneutic listening first, and that is a fundamental assumption of this paper.

The work of the QUASAR Project (Stein, Smith, Henningsen, & Silver, 2000) has proved helpful in analyzing the effects of discourse on learning mathematics, and in particular on measuring the *cognitive demand* of the mathematical task at hand. Teacher discourse can influence both the amount and the quality of learning that takes place, and may often inadvertently lower the level of the mathematical task from cognitively demanding to one of rote application of procedures. When students learn according to the higher levels of Bloom's taxonomy (Bloom, 1956), they are better equipped to advance to the next level of mathematical complexity. But again, as with Sfard, this type of analysis focuses on the resulting effects of the discourse, rather than contributing directly to a morphology of mathematical discourse types. A thorough analysis of how these discourse types might be associated with the cognitive demand of various mathematical tasks on students may prove extremely insightful. This awaits a future study.

Morphology of Teacher Discourse

Through our extensive observations and analyses of many hours of classroom teacher discourse, we have identified five categories into which teacher talk seems to naturally fall. This teasing out of categories has proved helpful in that it allows us to describe, characterize and contrast teacher mathematical discourse. We first list these five categories, and follow this brief introduction with a more complete description. The first category, (i) is the big ideas – "*math talk*" and norm setting - that underlie and permeate all teacher mathematical discourse (Cobb, Yackel, & McClain, 2000). The modeling of mathematical talk is what one sees when using the "big picture" or wide-angle lens to study teacher mathematical discourse. The rules for student engagement and interaction with mathematics are laid out in this way. The second category, (ii), is evident by focusing on the *affective* or emotional aspect to mathematical discourse, whereby social and socio-mathematical norms are further established in the classroom, and through which students are motivated to be successful learners (Lerman 2001, Cobb, Yackel, & McClain 2000). Narrowing the view somewhat, we focus more closely on the components of discourse that contribute to making mathematics happen in the classroom (iii). These are the discursive tools, examples and operations relating to *content* that the teacher employs to help the student achieve and learn the intended focus. (iv) Sometimes overlooked or underemphasized is another set of *tactical tools* that make learning happen among students and are at a meta-level in relation to the actual discursive tools used by both teachers and students engaged in mathematics study and learning. (v) *Ending discourse* is also meta- mathematical with the particular objective of reviewing the material previously introduced, and allowing students to display what they have learned. We elaborate on these categories here.

i) Big ideas

Throughout all mathematics teacher discourse, two overarching big ideas are addressed, (grouped together here) even though at times the teacher may be unaware of their centrality. The teacher's act of engaging in "math talk" provides students with a modeling of general patterns of discourse in mathematics. Students learn from their teachers' discourse how to talk about and think about mathematics, and are at the same time tacitly exposed to their teachers' epistemological and ontological views of mathematics (Thompson, 1984; Ernest, 1988; Lerman, 1983). When understood by the receptive student, they are typically conveyed tacitly, without any explicit practice or discussion. For example, the extent to which a teacher thinks that mathematics is discovered truth, found in the imagination of

intelligent people, as opposed to a newly invented language with vocabulary, syntax, symbols, grammar and aesthetic considerations, is conveyed through teacher discourse. Even without trying, the teacher sets the example in the classroom for how students will talk and think about mathematics, and for how mathematics may be connected to other curricula and to the world (Frade, 2005; Ernest, 1992).

Secondly, intertwined with this action of modeling mathematical talk, is the notion of norm setting, both social and socio-mathematical in nature. As part of teacher discourse, both tacitly and explicitly, the characteristic norms for the teacher's classroom are established – how to conduct oneself in the classroom, standards of neatness for student work, how much work to show, what is acceptable behavior and what is not, what work ethic is expected, and what exactly is required as an answer or justification in a mathematical task – acceptable "proof talk". These norms are established both explicitly and tacitly by the teacher modeling mathematical talk, demonstrating through the use of language (both verbal and non-verbal) the norms in the classroom, and discussing, even negotiating classroom norms directly with her students. Whenever norms are conveyed tacitly there is the danger that students will not understand them, and the teacher may make the mistake of assuming students understand when they do not. Establishment of all these desired norms does not just happen without deliberate action on the part of the teacher. If the reflective teacher finds that students are repeatedly making the same kinds of mistakes in their answers, then the teacher should consider spending more time on this category of discourse in the classroom, making the norms explicit rather than tacit.

ii) Affective discourse

As an undercurrent to and a major component of the setting of social and socio-mathematical norms in the mathematics classroom are the affective aspects of teacher discourse (e.g., Presmeg & Balderas-Canas, 2001; Goldin, 2000; Schoenfeld, 1998, 1999, 2002). These include animated teacher discourse that inspires students in their work, motivates them to work hard and achieve results and enjoy doing mathematics, praise for students in their efforts when such achievements are attained, affirmation and rewards for positive results and progress, inclusion of all students by being aware of status issues in the classroom (Knott, 2007), and being sensitive to the diverse needs (learning styles) of students. Inspiring students to want to learn and showing them how to enjoy mathematics may be the single greatest challenge left for mathematics teachers. Frustration and discouragement is frequent in mathematics classrooms so this becomes a very important piece of practice.

This affective (positive) nature of discourse is achieved both explicitly through teacher discourse, encouragement and counseling with students, and tacitly through the use of body language (voice inflection and smiling) and gestures. Much work has been done on the affective nature of mathematics, often its negative aspects – how students learn to dislike, avoid, even despise mathematics, but the focus has not been on changing discourse itself nor on how teachers can effectively reverse this trend (Boaler, 2002a, 2002b; Forgasz & Leder, 1996; Zevenbergen, 2002).

iii) Discursive mathematical tools

All teachers know that learning in any subject is the responsibility of the student, but the teacher is ultimately responsible for clarifying both the technical and abstract aspects of mathematics so learning can happen in the classroom. This is achieved by maintaining a competent level of mathematical discourse, including but not limited to such competencies as proving, conjecturing, validating, justifying, generalizing, explaining, repeating, and performing. These explicitly mathematical competencies are influenced by teachers' ways of knowing, understanding and talking about mathematics. For example; how many different ways of solving a problem will the teacher accept?

The term "discursive" is used here because it implies the importance of proper sequencing of mathematical ideas. Teachers must apply this sequencing tool on a daily basis, to make sure their explanations are not over the heads of their students. Another element, or discursive tool, is that of making learning goals, strategies, proficiency standards and achievement expectations explicit. When teachers aren't engaging directly and explicitly in mathematical examples with students, the teacher will engage in a subtle discourse that encourages discovery and enables students to be mathematically creative and discursive themselves. Here, the teacher employs tactical discursive tools.

iv) Tactical tools to promote learning

The teacher has a wide range of discourse moves that are meta-mathematical in nature, meaning that they do not directly supply mathematical content but rather they are *about* mathematics, and employ the language or "register" of mathematics. These moves may be more important in making mathematics learning happen in the classroom than the discursive content moves, by encouraging students to participate in shaping their own learning. In particular, the collective act of abstraction that occurs for students during the course of a lesson is facilitated and choreographed by teacher meta-mathematical discourse.

The process by which students eventually arrive at an understanding depends on discourse by and with the teacher and with other students that is necessarily social.

According to Cobb, Boufi, McClain, & Whitenack (1997) collective reflection occurs as students participate in reflective discourse. Teachers play a huge role in effecting this. They must use input language (giving instructions and explanations) that is both comprehensible and appropriate. To be effective, during the course of a mathematics lesson teachers must employ many meta-mathematical moves (Krussel, Edwards, & Springer, 2004): steering, probing, redirecting, clarifying, validating, prompting, rephrasing, re-voicing, responding with a question, generalizing, organizing, or even at appropriate times maintaining silence and actively listening, both interpretively and hermeneutically.

These are some of the many tactical moves a teacher must have on hand, and use habitually, during any lesson. This is not directly mathematical content discourse per se, since it is not discourse that is used in direct engagement with a mathematics problem. It is discourse that is one level removed, that occurs between teacher and students as the students engage in the activities. All of this requires considerable wisdom and judgment on the part of the teacher. As Cobb quotes Bauersfeld (Cobb, Boufi, McClain, & Whitenack, 1997) "… the core of what is learned through participation is when to do what and how to do it.…the core part of school mathematics enculturation comes into effect on the meta–level and is 'learned' indirectly." (p. 7).

The use of the term 'meta-mathematics' by mathematicians and philosophers relates to discussions in the philosophy of mathematics. In these cases experts are justifiably arguing fine points of interest relating to the character of mathematics, e.g., the necessity for elegance in writing proofs. But in a broader sense, the tactics used in the classroom that guide the development of mathematical learning, the input language, often pedagogical in nature, that is outside the realm of mathematics is properly identified as meta-mathematical.

v) Ending discourse

There is sufficient distinction in the discourse that takes place at the end of the lesson that it deserves its own category. Here, teacher discourse focuses on closing the lesson in such a way that students learn from the experience. This ending discourse is critical in the learning process. It includes such aspects as summing up, bringing together, synthesizing, making connections, assessing learning gains, looking back and reflecting, asking "what did you learn?". Of course assessment is an on-going process and is connected to the kind of listening advocated here, namely interpretive or hermeneutic. Formal assessments can also be used as teaching opportunities; these are certainly affective experiences for most students, too often with negative results. If teachers can learn to create tests worth taking (Wiggins, 1992) that empower students and give them an opportunity to really

display what they have learned and achieved, this will be a new form of productive, ending discourse.

The reader is asked to keep these five prototypes in mind as we move through the rather long example transcript that follows, from a 9th grade public school classroom of one of the authors. In a later section we contrast these with categories in the analysis of *student* discourse.

A Classroom Vignette

[1] T: How do we determine whether one point is between two other points?

S1: You plot the points and see where they are on the real number line.

S2: Couldn't we use the mid-point formula?

S1: But you can't be sure that this point in exactly in the middle of the two other points.

[2] T: Don't we need some information about the location of the points to use the midpoint formula?

S3: How can you be sure that the points are on the real line? Can't the points be outside the real line?

S2: Yeah, we use (*x, y*) co-ordinates to locate the points.

[3] T: Why don't we look at an example. What if we take the points P(3,2), Q(6,4) and R(9,6) and plot them?

S4: The points are on the line with slope 2/3.

[4] T: Okay, now how do we check whether Q is between P and R?

S1: Just use the plot and you see the point Q is right in between.

[5] T: What if you don't have graph paper and you can't plot the points?

S2: You can just visualize it in your head.

[6] T: Can we use any formulas we learned?

S4: The mid-point formula?

[7] T: But you can't always be sure that one point will always be between the other two.

S5: Why don't we look at distances between the three points?

[8] T: That's a good idea. Does anybody remember how we calculate distances?

S6: The distance formula.

[9] T: Yes, but how can we use the distance formula to decide that Q is between P and R?

S6: Calculate PQ, then QR, and then see if they add up to PR.

[10] T: Does anybody disagree with this idea?

S4: Yeah, but does it always work?

S6: I think it does. You can check it on the real line if you like.

[11] T: If it works on the real line, do you think it works on every line?

S6: Yeah, cause the real line is just another line with no slope.

[Students perform calculations and determine that PQ+ QR = PR]

[12] T: Can we define between-ness now?

S6: We already did. Just calculate the three distances and see if the two smaller ones add up to the total distance.

[13] T: Okay, so we say that a point B is between A and C if AB+ BC = AC, and to make life easier we write A-B-C. Now the question is does it work the same way in the taxicab world?

S5: But how are we going to calculate distances there? Don't we need that?

S7: You can just count on a graph paper.

[14] T: Can we come up with a formula maybe? Just like the distance formula we already know? $d = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

[Silence approximately 1 minute]

[15] T: Does anyone remember how we came up with the normal distance formula? Does Student 9 remember this?

S9: We like used Pythagoras. You know, like make a right triangle and then see what the length on the hypotenuse is.

[16] T: So how would it work on the co-ordinates we have plotted on the board? Can someone show the class?

[Students volunteer to demonstrate]

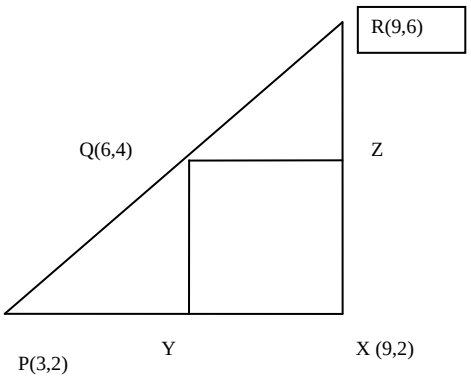

Figure 1. Drawing on the board

S6: So, we can draw a horizontal line that goes across from P this way (indicating PX), and then a vertical line from here to R (indicating XR).

[Note: At this stage point X has not yet been labeled]

[17] T: Okay, that is good. Should we also label this point where the line starts moving up (indicating intersection of PX and XR]?

S4, 8: (9,2)

[18] T: Let's call this point X. Now how do we calculate the distance between P and R?

S6: Why can't we just plug the co-ordinates into the formula, instead of having this new point?

S2: Yeah, I already figured it was root 52.

[19] T: Why don't we do the distance both ways. You can use the distance formula and then use the right triangle PXR

[20] *[Teacher walks around the room. Several students ask clarification for what the second way is. Teacher asks to use the Pythagorean Theorem on the right triangle.]*

[21] T: So, what did you get?

S2: root 52

[Other students state they got the same answer]

[22] T: Now how did using the Pythagorean Theorem connect to the distance formula?

S8: It was like the same thing. Cause you get the same numbers in the root formula.

[23] T: Now would this be the same answer if we had to walk from P to R in a city where there are blocks?

S2: Yeah, the answer would be the same.

[24] T: Why don't you talk about this to your neighbors and see if the answer is the same?

[25] *[5 minutes pass]*

[26] T: Is the answer the same? What did you come up with [looking at group of students]?

S8, 11, 12: We thought it would have to be different cause you can't really go across from P to R unless there is a field there or you're jaywalking. So, if you were walking along blocks and crossing where you're supposed to, then you'd have to go from P to X and then from X to R. So it is 10.

S6: You could also go a different route and get 10.

[27] T: What do you mean?

S6: You could like do a zigzag thing. Walk two blocks east from P, then go north, then go east again and then go up.

[28] Teacher labels new points on the board [Y and Z]

[Classroom discussion eventually leads to the consensus that distance in the taxicab world is calculated by counting the number of blocks traveled either east-west plus the number of blocks traveled up-down]

[29] T: We can write this formula as $d_T = |(x_1-x_2)| + |(y_1-y_2)|$ and we'll use d_E for the normal way of calculating distance. Now can we check whether Q is between P and R?

S8: Do we do the same thing like before?

[30] T: What do you mean?

S8: Like see if those three distances add up?

[31] T: What does the class think?

S6: I think it will be the same.

[32] T: Same what?

S6: Like the same rule, you know AB+ BC has to equal AC.

[Calculations reveal that between-ness does work out the same way]

S7: But what about these other points….can't we go from P to R like in a city, where you are trying to avoid a block. What I'm saying is can't you go through a different point, like some (other) point X and get to R. Does that mean there are other points P and R that are between but not on the line?

[33] T: You mean that the addition works out but these other points like X, Y and Z are really not on the line PR?

[This elenchus (refutation) led to a discussion of the difference between "metric" between-ness and between-ness as defined in Euclidean Geometry. The discussion led us to re-examine the Euclidean hypothesis for between-ness and reach the following conclusion]

[Classroom is loud and students talk about whether these other points X,Y and Z are also "between" P and R in the new metric]

S8: All the distances add up the way they should. So they are between although they aren't on the same line PR.

[34] T: We will impose the requirement in the definition of betweenness that points be on a particular line to take care of this problem of "metric" between-ness in Taxicab geometry. This way we can use the same definition in both geometries.

Analysis of Discourse

A first reading of the teacher discourse in this vignette reveals that both the teacher and the students are asking questions. Clearly this is not a mathematics classroom where the teacher does all the telling. The norm has been established that the students themselves are responsible for finding answers to their questions, and for asking questions in addition to those posed by their teacher. The teacher works as a tactical guide to gently choreograph by steering, probing, and redirecting their sometimes incorrect and unfocused thoughts. One big idea here is that students will work together in groups to explore a new mathematical idea, to hypothesize and conjecture, try examples, and to make connections to existing knowledge, ultimately reaching consensus. In this case they arrive at a definition for the concept of collinearity in this non-Euclidean metric.

An analysis of the types of questioning employed by the teacher reveals that the majority of his questions are 'why?' or 'what if?' or 'how about?', open-ended questions. Although some of his questions may be construed as 'yes/no' questions, they are more typically of the form 'can we?', and the students clearly understand the norm that this type of questioning means they must think further about the subject, rather than just reply with a simple yes or no. Like examples of discourse from this excerpt are grouped, and then our conclusions are presented.

[i] Big ideas

In this classroom, the teacher has previously established the classroom norm that teacher questions are invitations to further investigation. This is different than the norm in classrooms where the teacher is the voice of authority and the supplier of final answers, and where their questions serve to elicit preconceived final answers. Throughout, the teacher is carefully crafting his questions and suggestions towards a particular goal – that of a working definition for and understanding of the concept of between-ness in different situations.

[ii] Affective discourse

There is little affective discourse in this excerpt. In [7] the teacher makes a statement intended to generate some cognitive dissonance (this is a gentle way of saying possibly they are wrong) in the students, thus causing them to rethink their idea of using the midpoint formula. In [8], the teacher provides some affective reinforcement by affirming the appropriateness of looking at the distance between points, and asks students to recall another formula for calculating distances. In [17], the teacher affirms the student's work at the board.

[iii] Discursive mathematical tools

In [1], the teacher first poses an initial mathematical, discursive question and thereby sets the stage for the unfolding vignette. In [6], the teacher directs the discourse in an anticipatory fashion by asking the students to think about the mathematically relevant formulas they have just studied, thus appealing to them to make connections to prior knowledge. This has the effect of redirecting the intended focus of the discourse towards a formula. In [11], the teacher encourages the students to extend and generalize their thinking by asking them to consider whether this process works on every line in the plane. He then asks the students to again think about extending their definition to this new situation.

In [14] the teacher again pushes the students towards coming up with a formula. In [15], the teacher provides the students with the necessary connection to their prior knowledge by asking if they remember the normal distance formula. This sort of intervention is critical. Since the teacher had a clear idea of where the discourse should go, he was ready at the appropriate time – after an extended silence – with the critical piece of information that allowed the students to proceed towards the answer to this question. In [19] the teacher suggests calculating the distance in two ways. The students take a few minutes to perform this calculation individually, with some students asking for clarification.

In [20], the teacher circulates in the room, evaluatively and hermeneutically listening to determine when all students are on board, and in [21], asks for their answer. Students compare and find they have the same answer. After assuring himself that all students have the same answer, the teacher again asks the students to extend their thinking to the taxicab (Manhattan) metric. In [29], the teacher sums up the students' ideas and puts their somewhat awkward words in the form of a symbolic formula. This is the extent of the discursive math talk on the part of the teacher throughout this excerpt.

[iv] Tactical moves to promote learning

In [2] the teacher steers the conversation back to the idea of using the midpoint formula by asking what is needed in order to use it. In [3], the teacher suggests that the students look at an example. In [4] the teacher redirects the discourse away from the concept of slope to the idea of between-ness, the intended focus of their investigation. In [5] the teacher again steers the students away from a graphical solution and towards an algebraic solution. In [9] the teacher further refocuses the students and probes their thinking by asking how this formula can be used to elicit the information needed to answer the initial question. In [13] the teacher revoices the students' inadequate and somewhat inarticulate voicing of the definition. Again the teacher is carefully anticipating the student's learning.

At every step the teacher is carefully guiding and steering the students towards the intended result. In no way can this discourse be considered a "go with the flow" affair, with the teacher simply following along behind the students and their thoughts going wherever they go with the problem. The teacher makes a deliberate effort to direct his students' progress on this problem.

In [16], the teacher asks for a volunteer from the class to demonstrate at the board so that the entire class is on the same wavelength, thus clarifying the task. This makes the intended focus explicit, becoming the attended focus for the entire class once it is put on the board. In [18], the teacher is refocusing the students' attention on the particular details of the work on the board and carefully steering them in the desired direction. The teacher, in [22], then asks the students to make connections between prior knowledge about the Pythagorean Theorem and this distance formula. After some students suggest that the answer would be the same, the teacher, knowing that it is not the same, doesn't tell the students that, but rather sends them [24] to their small groups or neighbors to discuss this idea. A five minute break in the discourse ensues, during which time students are busy discussing various ideas with their neighbors, while the teacher is again hermeneutically listening, an appropriate tactical silence [25]. In [26], the teacher calls on one of the small groups to share their findings with the whole class. Then at [27] the teacher makes a call for clarification.

[v] Ending discourse

In [10] the teacher is asking for consensus, a form of ending, among the students by asking if anybody disagrees. In this way, the teacher monitors the students' progress and ascertains that they share the same intended focus. In [12] the teacher now brings this section of the discourse to a close by asking the students to provide a definition for between-ness. Again, in [30], the teacher again appeals for clarification and [31] is a call for consensus. Once the students reach consensus about the sameness of the distance formula with the taxicab metric there ensues a discussion about the interpretation of between-ness in this new metric. The students discuss the difference between "metric" between-ness using the distance formula and the usual Euclidean hypothesis for between-ness. The students engage in a loud discussion about these ideas, again reaching consensus in the idea that these points are in fact between, even though they are not on the same Euclidean straight line. In [33 and 34] the teacher summarizes the entire episode and makes the declaration that points must be on a particular line in order to talk about between-ness so that there is some consistency between the concepts in the two metrics.

Notice that the teacher makes few mathematical statements throughout the lesson, does no teaching in the traditional sense of imparting mathematical material throughout the episode, until [29] and again in [34] in the last utterance in which he brings closure to the discussion and defines the new-found rules. But this occurs only after extended discourse and exploration by the students. This discourse was chosen because the teaching taking place is not traditional, and yet it is evident that a great deal of learning on the part of the students is taking place.

It is clear from this grouping of the types of discourse occurring in this excerpt that the bulk of the teacher discourse is of the tactical form, type (iv). On close analysis, it is apparent that this teacher engaged in very little direct instruction, as is evident by the small number of examples of type (iii) discourse.

Student Discourse

The work of Weaver, Dick, and Rigelman (2005) has established that student discourse is distinct from teacher discourse. The various *modes* of student discourse are laid out in their work, followed by a discussion and examples of *types* of student mathematical discourse. Lastly, the *tools* for student discourse are listed. This discussion serves as a platform from which to emphasize the differences between student and teacher mathematical discourse (see also Krussel, Dick, & Springer, 2004). Knowledge of these differences also makes it possible for teachers to elicit a much more rich and complex kind of discourse from their students. A brief description of the modes, types and tools of student discourse, as established by Weaver et al, follows.

Modes of student discourse

Student mathematical discourse—that is, the act of articulating mathematical ideas or procedures—may take place in several modes, for example student to teacher, student to student, student to group or class, or individual reflection.

Types of student discourse

Effective mathematical discourse is an iterative process by which students and teachers engage in a variety of types of discourse at different cognitive levels. Student questions lead to explanations and justifications that may be challenged and subsequently defended, which might in turn lead to the formation of new generalizations or conjectures, thereby initiating a new cycle. Examples of these types of discourse include answering, questioning, explaining, justifying, challenging, defending, sharing an observation or prediction, generalizing and conjecturing.

Tools for student discourse

Students and teachers may employ a variety of tools to help them communicate the mathematical ideas or procedures. The tools they choose to use are important indicators of their level of sophistication with respect to mathematics. Tools may be verbal, gesturing or acting, written, graphs, charts, sketches, manipulatives, symbolization, notation, and computers or calculators

Distinctions between Teacher and Student Discourse

It seems clear from the above analysis that the most distinctive and frequent aspect of teacher discourse in this episode is the tactical, or meta-mathematical discourse. One of the biggest differences readily apparent in any classroom observation is the size of an experienced teacher's discourse toolbox and the facility with which the teacher chooses and uses tools. The teacher has presumably all of those that are available to the student (primarily discursive) and more. As Sfard (2000b) suggests, the focus of teacher discourse shifts back and forth among the intended, attended and pronounced focus (the objective of the lesson). The teacher is constantly listening to and evaluating the discursive process for evidence of student engagement, understanding and learning. To do this, a teacher must engage in this meta-mathematical discourse, and employ tactical moves to advance students' thinking and learning.

By contrast, a student's mathematical discourse will spring from a rudimentary toolkit, with little or no overlap or connections among them (see for example the notion of concept image in Vinner & Dreyfus, 1989). The experienced teacher moves easily among her tools, selecting the appropriate one for the immediate task.

Characterizing teachers' discourse as a different species in opposition to students' discourse, illustrates the difference in the complexity between the two types of discourse. The most important difference between the student and teacher discourse is that teachers must have in their toolbox a working meta-mathematical vocabulary and choices to guide the tactical success of the lesson. This allows the teacher to talk about doing mathematics with students, to create opportunities to steer students' attention in a particular direction. It is the role of the teacher to encourage students to pursue the problem at hand as well as other practical applications of mathematics, and help build the connections and provide the motivation that will ultimately inspire students to develop and use larger toolboxes. This meta-mathematical discourse as well as the general norm setting "math talk" helps students to move more freely about in an expanding toolbox.

By utilizing tactical discourse tools, as opposed to simply delivering content in lecture format and working through examples, the teacher has a much greater impact on the classroom environment and this clearly distinguishes it from student discourse. Again, examples of these meta-mathematical discourse moves are *probing, steering, re-directing, clarifying, validating, prompting, rephrasing, re-voicing, questioning, listening* and *maintaining silence*.

A teacher *probes* a student or group of students to elicit prior knowledge at the beginning of every class, and whenever that would prove useful and pertinent to the task at hand, especially if and when she feels that a student or group of students has failed to recognize its applicability. A teacher *steers* the mathematical discourse in a particular direction based on students' engagement in and demonstrated level of understanding of the task at hand, and based on the teacher's knowledge of where the task is going and where she wants it to end up. (Where is the ball of the focus bouncing?) Her steering move may also be based on her knowledge of each student's level of development and general mathematical competence.

She will *redirect* students' mathematical discourse if and when she feels that they have gone astray and have been distracted by irrelevant or incorrect mathematical assumptions. If however, she feels that students are on the right track, she may ask for *validation* of their mathematics. She may do this by *rephrasing* or *re-voicing* their mathematics so that they hear it in her words and have an opportunity to decide whether it is correct or not. Additionally, she may *question* or encourage her students' continued engagement by *maintaining* a timely *silence*, all the while *listening* to their talk.

All of these meta-mathematical moves require skill and practice on the part of the teacher, and are discourse moves that are unique to the role of the teacher. Students may ask questions of each other and/or their teacher, but they are generally content questions, or discursive questions in nature. Clearly, these discourse moves influence and direct student learning to the intended focus which is of course the objective of the lesson.

Summary

This morphology clearly illustrates that the toolbox the teacher brings to class each day is of a substantially different species than that borne by the student. The most significant difference between students' discourse and teachers' discourse is the employment of tactical or meta-mathematical moves that guide the course of the lesson and steer students towards effective learning. Teachers need to understand

the unique nature of these discourse moves and learn how to make use of them in the classroom.

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